

Optimal stopping of strong Markov processes

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Abstract

We characterize the value function and the optimal stopping time for a large class of optimal stopping problems where the underlying process to be stopped is a fairly general Markov process. The main result is inspired by recent findings for Lévy processes obtained essentially via the Wiener-Hopf factorization. The main ingredient in our approach is the representation of the β -excessive functions as expected suprema. A variety of examples is given.

Keywords: optimal stopping problem, Markov processes, Hunt processes, Lévy processes, supremum representation for excessive functions

1 Introduction

Consider a real-valued strong Markov process $X = (X_t)_{t \geq 0}$. Let \mathbb{P}_x and \mathbb{E}_x stand for the probability and the expectation, respectively, associated with X when initiated from x . The natural filtration generated by X is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and \mathcal{M} is the set of all stopping times with respect to \mathcal{F} .

We are interested in studying the following optimal stopping problem: Find a function V (value function) and a stopping time τ^* (optimal stopping time) such that

$$V(x) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_x \left(e^{-\beta \tau} G(X_\tau) \right) = \mathbb{E}_x \left(e^{-\beta \tau^*} G(X_{\tau^*}) \right), \quad (*)$$

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where the function G (reward function) is assumed to have some regularity properties to be specified later. On $\{\tau = \infty\}$ we define

$$e^{-\beta\tau}G(X_\tau) = \limsup_{t \rightarrow \infty} e^{-\beta t}G(X_t).$$

Recall that a non-negative, measurable function u is called β -excessive for X if the following two conditions hold:

$$\mathbb{E}_x\left(e^{-\beta t}u(X_t)\right) \leq u(x) \quad \forall t \geq 0, x \in S,$$

$$\lim_{t \rightarrow 0} \mathbb{E}_x\left(e^{-\beta t}u(X_t)\right) = u(x) \quad \forall x \in S.$$

A basic result, see Theorem 1 p. 124 in Shiriyayev [25], is that if the reward function G is lower semicontinuous then the value function V exists and is characterized as the smallest β -excessive majorant of G .

During the last decade the theory of optimal stopping for Lévy processes has been developed strongly. This research has been focused on analyzing the validity of the smooth pasting condition, see Alili and Kyprianou [1], Christensen and Irle [5], on the one hand, and solving particular problems, see Boyarchenko and Levendorskii [4], Mordecki [17], Novikov and Shiriyayev [19], [20], Kyprianou and Surya [15], Surya [26], Mordecki and Salminen [18], Deligiannidis, Le and Utev [9], on the other hand. The methodology to derive explicit solutions is in many cases based on the Wiener-Hopf factorization.

For random walks in discrete time optimal stopping problems with the reward functions $G(x) = x^+$ and $G(x) = (e^x - 1)^+$ were solved already by Darling et al. in [8]. In Mordecki [17] the results in [8] were lifted to the framework of Lévy processes with the focus to the reward functions $G(x) = (K - e^x)^+$ (put option) and $G(x) = (e^x - K)^+$ (call option). Furthermore, for the value function $G(x) = (x^+)^{\nu}$, $\nu > 0$, the problem was solved by Novikov and Shiriyayev in [19] and [20] for random walks and Lévy processes using Appell polynomials and Appell functions, see also Kyprianou and Surya [15] and Alili and Kyprianou [1]. In addition to the Wiener-Hopf factorization, the fluctuation identities constitute an essential tool in these solutions.

This technique was generalized under certain conditions to be applicable for more general reward functions independently in Surya [26] and Deligiannidis et al. [9]. In particular, in [26] it is assumed that there exists a function Q such that the reward function has the representation

$$G(x) = \mathbb{E}_0\left(Q(x + M_T)\right), \tag{1}$$

where T is an exponentially with parameter $\beta > 0$ distributed random variable independent of X and $M_t := \sup_{0 \leq s \leq t} X_s$ is the running maximum of

the Lévy process X . The function Q should have a unique positive root x^* such that $Q(x) < 0$ for $x < x^*$ and $Q(x) > 0$ and non-decreasing for $x > x^*$. Then the stopping problem (*) is, in fact, one-sided and the optimal stopping time is

$$\tau^* := \inf\{t : X_t \geq x^*\}.$$

Moreover, the value function is expressed in [26] as

$$V(x) = \mathbb{E}_0 \left(Q(x + M_T) 1_{\{x + M_T \geq x^*\}} \right).$$

In [9] the starting point is the assumption that there exists a function g such that

$$G(x) = \mathbb{E}_0 \left(g(x + X_T) \right). \quad (2)$$

To see the connection between Q in (1) and g in (2) recall the Wiener-Hopf factorization

$$X_T \stackrel{(d)}{=} M_T + I_T^\circ, \quad (3)$$

where $I_T^\circ \stackrel{(d)}{=} I_T := \inf_{0 \leq t \leq T} X_t$ and independent of X . Consequently,

$$Q(x) = \mathbb{E}_0 \left(g(x + I_T) \right). \quad (4)$$

Since in [9] the stopping problem is analyzed via the function

$$x \rightarrow \mathbb{E}_0(g(x + I_T))$$

the relationship (4) implies that the results in [26] and [9] are very close to each other.

Our main result characterizes the solution of the optimal stopping problem (*) similarly as in [19], [26] and [9] but is valid for very general strong Markov processes, e.g., for diffusions, Lévy processes and continuous time Markov chains. The main ingredient in our approach is the representation of β -excessive functions in the form

$$x \mapsto \mathbb{E}_x \left(\sup_{0 \leq t \leq T} f(X_t) \right)$$

for a function $f \geq 0$. This kind of representations have been studied in Föllmer and Knispel [11] and El Karoui and Meziou [10].

The structure of the paper is the following: In the next section we state and prove the main theorems which lead to explicit solutions to the optimal stopping problem (*) for general Hunt processes X and reward functions G with some representation properties. Depending on the form of the representing function \hat{f} the optimal stopping time is one-sided or two-sided. We furthermore present in section 2.2 a way for finding \hat{f} explicitly. In section 2.3 the connection to the representing measure approach is discussed. In Section 3 the results are illustrated with a variety of examples.

2 Main results

2.1 Verification theorems

We use the notation from the introduction and let $X = (X_t)_{t \geq 0}$ be a Hunt process taking values in a subset S of \mathbb{R} , e.g., an interval or a finite set. We refer to [18] for references and further discussions on Hunt processes. For a full account see Blumenthal and Gettoor [2], Chung and Walsh [7] and Sharpe [24]. In particular, X has the strong Markov property and is quasi left continuous with right continuous sample paths with left limits. For simplicity, we suppose that the lifetime of X is a.s. infinite.

Remark 2.1. *In some problems it is of interest to study X , e.g., up to the passage time of a level. In such cases X is viewed as a Hunt process with lifetime equal to the passage time. For instance, a Brownian motion X killed at the first passage time*

$$\zeta_0 := \inf\{t \geq 0 : X_t < 0\}$$

is a Hunt process with lifetime ζ_0 . It is not difficult to see how to refine our approach to cover these and more general lifetimes, and we leave this task to the interested reader.

Throughout the paper, the notation T is used for an exponentially distributed random variable assumed to be independent of X and having the mean $1/\beta$ where $\beta > 0$ is the discounting rate introduced in (*). We remark that a function u is β -excessive if it is 0-excessive, i.e, excessive, for X killed at T .

The following lemma is the key to our approach for the optimal stopping problem (*). It is proved similarly as Proposition 2.1 in [11]. We give a short proof for the sake of completeness.

Lemma 2.2. *Let $f : S \mapsto \mathbb{R}_+$ be an upper semicontinuous function and define $u(x) := \mathbb{E}_x\left(\sup_{0 \leq t \leq T} f(X_t)\right)$. Then the function $u : S \mapsto \mathbb{R} \cup \{+\infty\}$ is β -excessive.*

Proof. By the upper semicontinuity of f and the right continuity of the sample paths of X the function $t \mapsto \sup_{0 \leq s \leq t} f(X_s)$ is right continuous and,

hence, measurable. It follows that u is well-defined. Consider for $t > 0$

$$\begin{aligned}
u(x) &= \mathbb{E}_x \left(\sup_{0 \leq s \leq T} f(X_s) \right) \\
&\geq \mathbb{E}_x \left(\sup_{t \leq s \leq T} f(X_s); t \leq T \right) = \mathbb{E}_x \left(\mathbb{E}_x \left(\sup_{t \leq s \leq T} f(X_s) | \mathcal{F}_t \right); t \leq T \right) \\
&= \mathbb{E}_x \left(\mathbb{E}_x \left(\sup_{t \leq s \leq T} f(X_s) | X_t \right); t \leq T \right) = \mathbb{E}_x \left(u(X_t); t \leq T \right) \\
&= \mathbb{E}_x \left(e^{-\beta t} u(X_t) \right),
\end{aligned}$$

and, therefore,

$$\mathbb{E}_x \left(u(X_t); t \leq T \right) \leq u(x).$$

Next,

$$\begin{aligned}
\lim_{t \rightarrow 0} \mathbb{E}_x \left(e^{-\beta t} u(X_t) \right) &= \lim_{t \rightarrow 0} \mathbb{E}_x \left(\sup_{t \leq s \leq T} f(X_s); t \leq T \right) \\
&= \mathbb{E}_x \left(\sup_{0 \leq s \leq T} f(X_s) \right) \\
&= u(x),
\end{aligned}$$

where the second equality holds by the monotone convergence. \square

Example 2.3. Consider a regular linear diffusion X on \mathbb{R} . Let $x \rightarrow u_\beta(x, y)$ be the β -resolvent kernel associated with X . As is well-known, the function $x \rightarrow u_\beta(x, y)$ is β -excessive for any fixed y . We find the representation $u_\beta(\cdot, y)$ as expected supremum. Recall that

$$\mathbb{E}_x \left(e^{-\beta H_y} \right) = \frac{u_\beta(x, y)}{u_\beta(y, y)} = \begin{cases} \frac{\psi_\beta(x)}{\psi_\beta(y)}, & \text{if } x \leq y, \\ \frac{\varphi_\beta(x)}{\varphi_\beta(y)}, & \text{if } x \geq y, \end{cases} \quad (5)$$

where $H_y := \inf\{t : X_t = y\}$ and ψ_β and φ_β are the fundamental increasing and decreasing, respectively, solutions for the generalized differential operator associated with X , see [3]. Introducing

$$x \rightarrow f_y(x) = \begin{cases} u_\beta(y, y), & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases}$$

we have by (5)

$$\begin{aligned}
\mathbb{E}_x \left(\sup_{t \leq T} f_y(X_t) \right) &= u_\beta(y, y) \mathbb{P}_x(H_y < T) \\
&= u_\beta(y, y) \mathbb{E}_x(e^{-\beta H_y}) \\
&= u_\beta(x, y).
\end{aligned}$$

For another example, consider the β -excessive function $\psi_\beta \wedge 1$. We assume that there exists a point y such that $\psi_\beta(y) = 1$. Notice that $\psi_\beta(x) \geq 1$ for all $x \geq y$ since ψ_β is non-decreasing. Define

$$x \rightarrow f_y^\uparrow(x) = \mathbf{1}_{[y, +\infty)}(x).$$

Then, calculating as above, we have

$$\mathbb{E}_x \left(\sup_{t \leq T} f_y^\uparrow(X_t) \right) = \begin{cases} 1, & \text{if } x \geq y, \\ \psi_\beta(x), & \text{if } x \leq y. \end{cases}$$

We consider now the optimal stopping problem (*) and assume throughout the paper that the reward function G is non-negative, lower semicontinuous and satisfies

$$\mathbb{E}_x \left(\sup_{t \geq 0} e^{-\beta t} G(X_t) \right) < \infty \quad (6)$$

Our main result which characterizes solutions of (*) is as follows:

Theorem 2.4. (*One-sided case*). Assume that there exists an upper semicontinuous function $\hat{f} : S \mapsto \mathbb{R}$ and a point $x^* \in S$ such that

- (a) (i) $\hat{f}(x) \leq 0$ for $x \leq x^*$,
- (ii) $\hat{f}(x)$ is positive and non-decreasing for $x > x^*$.
- (b) (i) $\mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \right) = G(x)$ for $x \geq x^*$
- (ii) $\mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \right) \geq G(x)$ for $x \leq x^*$.

Then the value function of the optimal stopping problem (*) is given by

$$V(x) = \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \mathbf{1}_{\{X_t \geq x^*\}} \right) = \mathbb{E}_x \left(\hat{f}(M_T) \mathbf{1}_{\{M_T \geq x^*\}} \right) \quad (7)$$

and

$$\tau^* := \inf\{t \geq 0 : X_t > x^*\}$$

is an optimal stopping time.

Proof. Since the function $x \mapsto \hat{f}(x) \mathbf{1}_{\{x \geq x^*\}}$ is non-decreasing, the second equality in (7) holds. Let \hat{V} denote the function given via these expressions, i.e.,

$$\hat{V}(x) := \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \mathbf{1}_{\{X_t \geq x^*\}} \right).$$

By Lemma 2.2, we know that \hat{V} is β -excessive, so that $(e^{-\beta t}\hat{V}(X_t))_{t \geq 0}$ is a non-negative càdlàg supermartingale (see [7] p. 101). Hence by optional sampling we have for all x and all stopping times τ

$$\hat{V}(x) \geq \mathbb{E}_x \left(e^{-\beta \tau} \hat{V}(X_\tau) \right). \quad (8)$$

By condition (a) we furthermore have $\hat{f}(x)1_{\{x \geq x^*\}} \geq \hat{f}(x)$ for all x , therefore using (b) we obtain from (8)

$$\hat{V}(x) \geq \mathbb{E}_x \left(e^{-\beta \tau} G(X_\tau) \right).$$

To conclude the proof we show that

$$\hat{V}(x) \leq \mathbb{E}_x \left(e^{-\beta \tau^*} G(X_{\tau^*}) \right). \quad (9)$$

The value function V being the smallest β -excessive majorant of G it then follows that $\hat{V} = V$ and τ^* is an optimal stopping time, as claimed. To prove (9) consider

$$\begin{aligned} \hat{V}(x) &= \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) 1_{\{X_t \geq x^*\}} \right) \\ &= \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) 1_{\{X_t \geq x^*\}}; \tau^* < T \right) \\ &= \mathbb{E}_x \left(\hat{f}(M_T) 1_{\{M_T \geq x^*\}}; \tau^* < T \right) \\ &= \mathbb{E}_x \left(\hat{f}(M_T); \tau^* < T \right) \\ &= \mathbb{E}_x \left(\hat{f} \left(\sup_{\tau^* \leq t \leq T} X_t \right); \tau^* < T \right) \\ &= \mathbb{E}_x \left(e^{-\beta \tau^*} \mathbb{E}_{X_{\tau^*}} \left(\hat{f} \left(\sup_{0 \leq t \leq T} X_t \right) \right) \right) \end{aligned}$$

by the strong Markov property. For the last equality note that by the memoryless property of the exponential distribution

$$\begin{aligned} &\mathbb{E}_x \left(\hat{f} \left(\sup_{\tau^* \leq t \leq T} X_t \right); \tau^* < T \right) \\ &= \int_0^\infty \mathbb{E}_x \left(\mathbb{E}_x \left(\hat{f} \left(\sup_{\tau^* \leq t \leq r} X_t \right) \middle| \mathcal{F}_{\tau^*} \right) 1_{\{\tau^* \leq r\}} \right) \mathbb{P}(T \in dr) \\ &= \mathbb{E}_x \left(\mathbb{E}_{X_{\tau^*}} \left(\int_0^\infty \hat{f} \left(\sup_{0 \leq t \leq r} X_t \right) \mathbb{P}(T \in dr) \right) e^{-\beta \tau^*} \right) \\ &= \mathbb{E}_x \left(e^{-\beta \tau^*} \mathbb{E}_{X_{\tau^*}} \left(\hat{f} \left(\sup_{0 \leq t \leq T} X_t \right) \right) \right). \end{aligned}$$

Since $\hat{f}(\sup_{0 \leq t \leq T} X_t) \leq \sup_{0 \leq t \leq T} \hat{f}(X_t)$ we obtain

$$\begin{aligned}\hat{V}(x) &\leq \mathbb{E}_x \left(e^{-\beta \tau^*} \mathbb{E}_{X_{\tau^*}} \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \right) \right) \\ &= \mathbb{E}_x \left(e^{-\beta \tau^*} G(X_{\tau^*}) \right),\end{aligned}$$

where we used (b)(ii) in the last step. This shows (9) and the claim is proved. \square

Remark 2.5. *By applying Theorem 2.4 to the process $-X$, the reward function $x \mapsto G(-x)$ and to the function $x \mapsto \hat{f}(-x)$ one obtains the analogous result for a threshold to the left applicable, e.g., to put-type options.*

Although one-sided stopping rules as in the previous theorem often arise naturally in many examples (especially for monotonic reward functions and discounting, see also [6] for a general discussion), for other problems a two-sided stopping rule is needed. The following theorem gives the corresponding result for this case.

Theorem 2.6. *(Two-sided case) Assume that there exists a lower semicontinuous function $\hat{f} : S \mapsto \mathbb{R}$ and points $x_*, x^* \in S$ such that*

- (a) (i) $\hat{f}(x) \leq 0$ for $x_* < x < x^*$,
- (ii) $\hat{f}(x)$ is non-increasing for $x < x_*$ and non-decreasing for $x > x^*$,
- (b) (i) $\mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \right) = G(x)$ for $x \notin [x_*, x^*]$,
- (ii) $\mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \right) \geq G(x)$ for $x \in [x_*, x^*]$.

Then the value function V of the optimal stopping problem () satisfies*

$$\begin{aligned}V(x) &= \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) 1_{\{X_t \notin [x_*, x^*]\}} \right) \\ &= \mathbb{E}_x \left(\hat{f}(I_T) 1_{\{I_T \leq x_*\}} \vee \hat{f}(M_T) 1_{\{M_T \geq x^*\}} \right)\end{aligned}\tag{10}$$

and

$$\tau^* := \inf \{t \geq 0 : X_t \notin [x_*, x^*]\}$$

is an optimal stopping time.

Proof. This theorem is proved similarly as Theorem 2.4. The second equality in (10) follows from the monotonic properties of \hat{f} . Letting

$$\hat{V}(x) := \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) 1_{\{X_t \notin [x_*, x^*]\}} \right)$$

it is seen from Lemma 2.2 that \hat{V} is β -excessive and, by conditions (a) and (b),

$$\hat{V}(x) \geq \mathbb{E}_x \left(e^{-\beta\tau} G(X_\tau) \right)$$

for all stopping times $\tau \in \mathcal{M}$. It remains to show that

$$\hat{V}(x) \leq \mathbb{E}_x \left(e^{-\beta\tau^*} G(X_{\tau^*}) \right). \quad (11)$$

For this, consider

$$\begin{aligned} \hat{V}(x) &= \mathbb{E}_x \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) 1_{\{X_t \geq x^*\}} \vee \sup_{0 \leq t \leq T} \hat{f}(X_t) 1_{\{X_t \leq x_*\}}; \tau^* < T \right) \\ &= \mathbb{E}_x \left(\hat{f}(M_T) 1_{\{M_T \geq x^*\}} \vee \hat{f}(I_T) 1_{\{I_T \leq x_*\}}; \tau^* < T \right) \\ &= \mathbb{E}_x \left(\hat{f} \left(\sup_{\tau^* \leq t \leq T} X_t \right) \vee \hat{f} \left(\inf_{\tau^* \leq t \leq T} X_t \right); \tau^* < T \right) \\ &= \mathbb{E}_x \left(e^{-\beta\tau^*} \mathbb{E}_{X_{\tau^*}} \left(\hat{f} \left(\sup_{0 \leq t \leq T} X_t \right) \right) \vee \left(\hat{f} \left(\inf_{0 \leq t \leq T} X_t \right) \right) \right), \end{aligned}$$

where the last equality is obtained by the strong Markov property. Since

$$\hat{f} \left(\sup_{0 \leq t \leq T} X_t \right) \vee \hat{f} \left(\inf_{0 \leq t \leq T} X_t \right) \leq \sup_{0 \leq t \leq T} \hat{f}(X_t),$$

we have

$$\begin{aligned} \hat{V}(x) &\leq \mathbb{E}_x \left(e^{-\beta\tau^*} \mathbb{E}_{X_{\tau^*}} \left(\sup_{0 \leq t \leq T} \hat{f}(X_t) \right) \right) \\ &= \mathbb{E}_x \left(e^{-\beta\tau^*} G(X_{\tau^*}) \right). \end{aligned}$$

Thus the proof is complete. \square

Remark 2.7. *The methodologies in [9] and [26] seem to be applicable only for one-sided problems (for Lévy processes). It is remarkable that the present approach provides a clean characterization also in two-sided cases (for general Hunt processes).*

2.2 Computing \hat{f}

To apply Theorem 2.4 and 2.6 for a given reward function G the question arises how to find a function \hat{f} fulfilling the conditions stated therein. For the related problem for underlying Lévy processes, Surya [26] gave a general way for sufficiently regular bounded functions G using the Fourier transform, but this seems to be specific for Lévy processes and is based on a certain assumption on the zeros of the Wiener-Hopf factors of X . We now discuss a more general way.

Let G be a given reward function and assume that for all x

$$\mathbb{E}_x\left(e^{-\beta t}G(X_t)\right) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (12)$$

Note that, by dominated convergence, (12) is fulfilled if in addition to (6) it also holds

$$\lim_{t \rightarrow \infty} e^{-\beta t}G(X_t) = 0 \quad \mathbb{P}_x\text{-a.s.}$$

Furthermore, assume that the function G is in the domain of the extended infinitesimal generator of X killed at T (cf. [21, VII.1]). Then, by definition, there exists a function \tilde{f} such that

$$\int_0^t |\tilde{f}(X_s)| ds < \infty \quad \mathbb{P}_x\text{-a.s.}$$

and the process

$$\left(e^{-\beta t}G(X_t) - G(X_0) + \int_0^t e^{-\beta s}\tilde{f}(X_s)ds\right)_{t \geq 0}$$

is a $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale. Consequently,

$$\mathbb{E}_x\left(e^{-\beta t}G(X_t)\right) - G(x) + \mathbb{E}_x\left(\int_0^t e^{-\beta s}\tilde{f}(X_s)ds\right) = 0,$$

and as $t \rightarrow \infty$ we obtain using (12)

$$G(x) = \lim_{t \rightarrow \infty} \mathbb{E}_x\left(\int_0^t e^{-\beta s}\tilde{f}(X_s)ds\right).$$

Hence, if, e.g.,

$$\mathbb{E}_x\left(\int_0^\infty e^{-\beta s}|\tilde{f}(X_s)|ds\right) < \infty,$$

the Lebesgue dominated convergence theorem yields

$$G(x) = \frac{1}{\beta} \mathbb{E}_x\left(\tilde{f}(X_T)\right). \quad (13)$$

Notice that for G in the domain of the infinitesimal generator A of X we have $\tilde{f} = -(A - \beta)G$. In this case equation (13) reflects one of the fundamental properties of the resolvent operator.

Remark 2.8. *For optimal stopping problems for Lévy processes it was assumed in [9] that a representation of type (13) (see also (2)) holds for G . Hence, the above considerations are useful when applying the methods in [9].*

Imitating the approach for optimal stopping of Lévy processes, we next express G as the expectation of a function of M_T (cf. (1) and (2)). Obviously, it holds

$$\mathbb{E}_x(\tilde{f}(X_T)) = \mathbb{E}_x(\mathbb{E}_x(\tilde{f}(X_T)|M_T)).$$

We assume that the regular conditional probability distribution

$$\mathbb{P}_x(X_T \in dy | M_T = z) := \mathbb{P}_x(X_T \in dy, M_T \in dz) / \mathbb{P}_x(M_T \in dz) \quad (14)$$

exists and has the needed regularity properties so that it holds

$$\frac{1}{\beta} \mathbb{E}_x(\tilde{f}(X_T)) = \mathbb{E}_x(Q(M_T; x)), \quad (15)$$

where

$$Q(z; x) := \frac{1}{\beta} \int_{-\infty}^z \tilde{f}(y) \mathbb{P}_x(X_T \in dy | M_T = z). \quad (16)$$

These observations are now applied to characterize solutions for optimal stopping problems for Lévy processes and linear diffusions via Theorem 2.4. Lévy processes are discussed first. The following lemma is well known. We present a short proof for the sake of completeness.

Lemma 2.9. *For a Lévy process X (which is not a subordinator or a compound Poisson process) it holds that*

$$\mathbb{P}_x(X_T \in dy | M_T = z) = \mathbb{P}_z(I_T \in dy). \quad (17)$$

In particular, the conditional distribution does not depend on x .

Proof. The claim follows from the Wiener-Hopf factorization (3) and the spatial homogeneity of X . Indeed,

$$\begin{aligned} \mathbb{P}_x(X_T \in dy, M_T \in dz) &= \mathbb{P}_0(x + X_T \in dy, x + M_T \in dz) \\ &= \mathbb{P}_0(x + M_T + I_T^\circ \in dy, x + M_T \in dz) \\ &= \mathbb{P}_0(z + I_T^\circ \in dy, x + M_T \in dz) \\ &= \mathbb{P}_z(I_T \in dy) \mathbb{P}_x(M_T \in dz). \end{aligned}$$

□

Proposition 2.10. *Consider the optimal stopping problem (*) in case of a Lévy process X (as introduced in Lemma 2.9). Assume that there exists a function \tilde{f} such that (13) holds and define*

$$Q(z) := \frac{1}{\beta} \int_{-\infty}^0 \tilde{f}(z+y) \mathbb{P}_0(I_T \in dy). \quad (18)$$

If Q fulfils (a) in Theorem 2.4 then the value function of the optimal stopping problem () is given as in (7) with $\hat{f} = Q \vee 0$.*

Proof. Since Q is assumed to fulfil (a) in Theorem 2.4 also \hat{f} fulfils this condition. We need to verify that \hat{f} satisfies (b) in Theorem 2.4. Definition (18) of Q , Lemma 2.9, and formula (15) yield

$$G(x) = \frac{1}{\beta} \mathbb{E}_x(\tilde{f}(X_T)) = \mathbb{E}_x(Q(M_T)).$$

Consequently, since $Q \leq \hat{f}$ we have for all x

$$G(x) = \mathbb{E}_x(Q(M_T)) \leq \mathbb{E}_x(\hat{f}(M_T)) \leq \mathbb{E}_x\left(\sup_{0 \leq t \leq T} \hat{f}(X_t)\right). \quad (19)$$

In case $x > x^*$ - since Q satisfies (a) in Theorem 2.4 - (19) holds with equalities instead of inequalities, and this completes the proof. \square

Next we consider the optimal stopping problem (*) for a regular linear diffusion. For simplicity, it is assumed that X is conservative and takes values on the whole \mathbb{R} . The following result is extracted from [3, II.19]. We use the notation from Example 2.3.

Lemma 2.11. *For a conservative regular linear diffusion X on \mathbb{R} it holds that*

$$\mathbb{P}_x(X_T \in dy | M_T = z) = \frac{\psi_\beta(y) m(dy)}{\int_{-\infty}^z \psi_\beta(u) m(du)}, \quad (20)$$

where m is the speed measure. In particular, the conditional distribution does not depend on x .

The next result can now be proved analogously as Proposition 2.10.

Proposition 2.12. *Consider the optimal stopping problem (*) for a diffusion X as introduced above. Assume that there exists a function \tilde{f} such that (13) holds and define*

$$Q(z) := \frac{1}{\beta} \frac{\int_{-\infty}^z \tilde{f}(y) \psi_\beta(y) m(dy)}{\int_{-\infty}^z \psi_\beta(y) m(dy)}. \quad (21)$$

If Q fulfils (a) in Theorem 2.4 then the value function of the optimal stopping problem () is given as in (7) with $\hat{f} = Q \vee 0$.*

To apply Proposition 2.12 it is useful to have condition on hand to guarantee that Q fulfills (a) in Theorem 2.4. The following Lemma gives such a condition in terms of the function \tilde{f} which is known explicitly in many situations of interest.

Lemma 2.13. *In the setting of Proposition 2.12 assume that there exists \tilde{x} such that $\tilde{f}(x) \leq 0$ for $x \leq \tilde{x}$ and $\tilde{f}(x)$ is non-decreasing for $x \geq \tilde{x}$. Then $Q \leq 0$ or there exists x^* such that Q fulfils (a) in Theorem 2.4.*

Proof. For simplicity we assume that the measure m is absolutely continuous, i.e. $m(dy) = m(y)dy$, and let

$$x^* = \inf\{z : \int_{-\infty}^z \tilde{f}(y)\psi_{\beta}(y) m(y)dy > 0\}.$$

If $x^* = \infty$, then the claim trivially holds. Therefore, we assume $x^* < \infty$. It remains to prove that Q is non-decreasing for $x > x^*$. Obviously $x^* > \tilde{x}$ and therefore by the assumption on \tilde{f} we get

$$\int_{-\infty}^z (\tilde{f}(z) - \tilde{f}(y))\psi_{\beta}(y) m(y)dy \geq 0.$$

Using this fact, by differentiating the function Q we immediately see that $\frac{d}{dx}Q(x) \geq 0$. \square

Remark 2.14. *It is critical that the function Q defined in (18) for Lévy processes and in (21) for diffusions does not depend on x . This clearly follows from the fact that the conditional distribution of X_T given M_T does not depend on the initial state of the process. Path decomposition theorems for strong Markov processes imply such results in general, see Millar [16]. This observation yields a method for finding \hat{f} for general underlying processes.*

2.3 Connection to the representing measure approach

Another approach to the solution of optimal stopping problems is the observation that finding the solution of such problems is equivalent to finding the Choquet-type integral representation of the value function in terms of the resolvent kernel $u_{\beta}(x, y)$. The Radon measure σ that appears in this representation characterizes the β -excessive value function V , and furthermore, the stopping region can be described as the support of σ .

This approach was first discussed in [22] for optimal stopping problems of diffusions. In [18] a verification theorem based on this approach applicable for general Hunt processes is presented. Moreover, in [18] the appearance of a function of the maximum in the solution of optimal stopping problems for

Lévy processes is explained via this approach and the Wiener-Hopf factorization.

The critical point to utilize the approach is to find a candidate solution for the measure σ . In [23], this was carried out explicitly for spectrally positive Lévy processes in the Novikov-Shiryaev problem in terms of the Appell polynomials of X_T using the Laplace transforms. In Proposition 2.15 we extend these results by pointing out a connection between this approach and our approach in this paper for a general Hunt process X which does not have negative jumps, that is,

$$\mathbb{P}_x\left(\inf_{t \geq 0}(X_t - X_{t-}) \geq 0\right) = 1 \quad \forall x \in S. \quad (22)$$

We work in the setting of Hunt processes satisfying the set of assumptions as given in [18, Section 2]. In particular, it is assumed that there exists a dual resolvent with respect to a duality measure m and that Hypothesis (B) from [13] holds. Moreover, we assume that the Laplace transform of the first hitting time $H_x := \inf\{t : X_t = x\}$ admits for all $x_0 < x$ the representation.

$$\mathbb{E}_x(e^{-\beta H_{x_0}}) = \frac{u_\beta(x, x_0)}{u_\beta(x_0, x_0)} \quad \forall x_0 < x \in S, \quad (23)$$

where $T_x = \inf\{t > 0 : X_t = x\}$. This representation is valid for a large class of processes. A sufficient condition is that (I) and (II) below hold true, see [2, Chapter V.3 and Exercise VI.4.18].

- (I) The mapping $x \mapsto \int_S u_\beta(x, y)f(y)m(dy)$ is continuous, where f is a bounded and measurable function with compact support.
- (II) Every point $x \in S$ is regular for $\{x\}$, that is

$$\mathbb{P}_x(T_x = 0) > 0, \quad (24)$$

where $T_x = \inf\{t > 0 : X_t = x\}$.

Assumption (I) is (4.1) in [2, p. 284], and assumption (II) implies that X has a local time at every point of the state space. Condition (23) furthermore holds true whenever X is a spectrally positive Lévy process, see [14, Theorem 3.12, Corollary 8.9].

Note that, in case X have positive jumps, overshoot occurs in the problem. Nonetheless it is possible to find an explicit representation of the measure σ .

Proposition 2.15. *Let X be a Hunt process as described above with no negative jumps that fulfills condition (23). Assume that for the optimal stopping*

problem (*) there exists a function \tilde{f} such that (13) holds. Moreover, assume that the function

$$Q(z) := \frac{1}{\beta} \int_{-\infty}^z \tilde{f}(y) \mathbb{P}(X_T \in dy | M_T = z)$$

fulfills (a) in Theorem 2.4. Then the value function V of (*) has the representation

$$V(x) = \int_{[x^*, \infty) \cap S} u_\beta(x, y) \sigma(dy) \quad \forall x \in S,$$

where the measure σ is given by

$$\sigma(dy) = \begin{cases} \tilde{f}(y)m(dy), & \text{for } y > x^*, \\ 0, & \text{for } y < x^*, \end{cases} \quad (25)$$

and

$$\sigma(\{x^*\}) = \frac{\mathbb{E}_{x^*}(\tilde{f}(X_T)1_{\{X_T \leq x^*\}})}{\beta u_\beta(x^*, x^*)}. \quad (26)$$

In the particular case that X is a spectrally positive Lévy process, then $\sigma(\{x^*\}) = 0$, and

$$V(x) = E_x(\tilde{f}(X_T)1_{\{X_T > x^*\}}).$$

Proof. We remark first that similarly as in the proof of Proposition 2.10 it can be verified that the value function V of the problem satisfies the equality

$$V(x) = \mathbb{E}_x(Q(M_T)1_{\{M_T \geq x^*\}}),$$

where x^* is the point associated with Q via (a) in Theorem 2.4. Now we have

$$\begin{aligned} V(x) &= \mathbb{E}_x(Q(M_T)1_{\{M_T \geq x^*\}}) = \frac{1}{\beta} \mathbb{E}_x\left(\mathbb{E}_x(\tilde{f}(X_T)|M_T)1_{\{M_T \geq x^*\}}\right) \\ &= \frac{1}{\beta} \mathbb{E}_x\left(\mathbb{E}_x(\tilde{f}(X_T)1_{\{M_T \geq x^*\}}|M_T)\right) = \frac{1}{\beta} \mathbb{E}_x(\tilde{f}(X_T)1_{\{M_T \geq x^*\}}) \\ &= \frac{1}{\beta} \mathbb{E}_x(\tilde{f}(X_T)1_{\{X_T > x^*\}}) + \frac{1}{\beta} \mathbb{E}_x(\tilde{f}(X_T)1_{\{M_T \geq x^*, X_T \leq x^*\}}). \end{aligned}$$

Since it is assumed that X moves continuously down we rewrite the second summand

$$\mathbb{E}_x(\tilde{f}(X_T)1_{\{M_T \geq x^*, X_T \leq x^*\}}) = \mathbb{E}_x(\tilde{f}(X_T)1_{\{T_{x^*} < T, X_T \leq x^*\}}),$$

and, by the strong Markov property,

$$\begin{aligned}\mathbb{E}_x \left(\tilde{f}(X_T) 1_{\{M_T \geq x^*, X_T \leq x^*\}} \right) &= \mathbb{E}_x \left(\mathbb{E}_x \left(\tilde{f}(X_T) 1_{\{X_T \leq x^*\}} | \mathcal{F}_{T_{x^*}} \right) 1_{\{T_{x^*} < T\}} \right) \\ &= \mathbb{E}_x \left(e^{-\beta T_{x^*}} \mathbb{E}_{X_{T_{x^*}}} \left(\tilde{f}(X_T) 1_{\{X_T \leq x^*\}} \right) \right) \\ &= \mathbb{E}_{x^*} \left(\tilde{f}(X_T) 1_{\{X_T \leq x^*\}} \right) \mathbb{E}_x \left(e^{-\beta T_{x^*}} \right).\end{aligned}$$

Letting $c = \mathbb{E}_{x^*} \left(\tilde{f}(X_T) 1_{\{X_T \leq x^*\}} \right)$ and using assumption (23) we obtain

$$\begin{aligned}V(x) &= \frac{1}{\beta} \mathbb{E}_x \left(\tilde{f}(X_T) 1_{\{X_T > x^*\}} \right) + \frac{1}{\beta} \mathbb{E}_x \left(\tilde{f}(X_T) 1_{\{M_T \geq x^*, X_T \leq x^*\}} \right) \\ &= \int_{(x^*, \infty) \cap S} \tilde{f}(y) u_\beta(x, y) m(dy) + \frac{c}{\beta u_\beta(x^*, x^*)} u_\beta(x, x^*) \\ &= \int_{[x^*, \infty) \cap S} u_\beta(x, y) \sigma(dy)\end{aligned}$$

with σ as in (25) and in (26). It remains to prove that σ is indeed non-negative. To this end, we use the fact that each β -excessive function u locally integrable with respect to the duality measure m can be decomposed uniquely in the form

$$u(x) = \int u_\beta(x, y) \sigma_u(dy) + h_u(x), \quad (27)$$

where σ_u is a non-negative Radon measure and h_u is a β -harmonic function, see [13, Theorem 2 p. 505 and Proposition 13.1 p. 523]. Since the value function V is β -excessive we may apply (27) to obtain

$$\int u_\beta(x, y) \sigma_V(dy) + h_V(x) = V(x) = \int_{[x^*, \infty) \cap S} u_\beta(x, y) \sigma(dy),$$

where σ_V is the non-negative Radon measure associated with V . By the Hahn decomposition we can write $\sigma - \sigma_V = \sigma^+ - \sigma^-$ for some non-negative measures σ^+ and σ^- . Therefore,

$$h_V(x) + \int u_\beta(x, y) \sigma^-(dy) = \int u_\beta(x, y) \sigma^+(dy).$$

By the uniqueness of the decomposition in (27) we obtain $h_V = 0$ and $\sigma^+ = \sigma^-$. Hence, $\sigma = \sigma_V$ is indeed a non-negative measure. This proves the first claim.

For spectrally positive Lévy processes it is well known that $-I_T$ is $Exp(\lambda)$ -distributed, where λ is the unique positive root of the equation

$$\mathbb{E}(e^{\lambda X_1}) = e^\beta.$$

Therefore, by the definition of Q and Proposition 17 we see that Q is continuous. In particular, $Q(x^*) = 0$. Using the Wiener-Hopf factorization (3) we obtain

$$\begin{aligned}\mathbb{E}_{x^*}(\tilde{f}(X_T)1_{\{X_T \leq x^*\}}) &= \mathbb{E}_0(\tilde{f}(x^* + M_T + I_T^\circ)1_{\{x^* + M_T + I_T^\circ \leq x^*\}}) \\ &= \int_0^\infty \mathbb{E}_0(\tilde{f}(x^* + t + I_T^\circ)1_{\{t + I_T^\circ \leq 0\}})\mathbb{P}(M_T \in dt).\end{aligned}$$

Therefore, $\mathbb{E}_{x^*}(\tilde{f}(X_T)1_{\{X_T < x^*\}}) = 0$, since

$$\begin{aligned}\mathbb{E}_0(\tilde{f}(x^* + t + I_T^\circ)1_{\{t + I_T^\circ \leq 0\}}) &= \int_{-\infty}^{-t} \tilde{f}(x^* + t + s)\lambda e^{\lambda s} ds \\ &= e^{-\lambda t} \int_{-\infty}^0 \tilde{f}(x^* + y)\lambda e^{\lambda y} dy \\ &= e^{-\lambda t} Q(x^*) = 0.\end{aligned}$$

Consequently,

$$\sigma(\{x^*\}) = \frac{\mathbb{E}_{x^*}(\tilde{f}(X_T)1_{\{X_T \leq x^*\}})}{\beta u_\beta(x^*, x^*)} = 0.$$

□

Remark 2.16. *By inspecting the previous proof one can see that assumption (23) can be weakened by assuming that (23) holds for $x_0 = x^*$ only.*

3 Examples

In this section we demonstrate the applicability of the above theory by solving some optimal stopping problems for diffusions, Lévy processes, and continuous time Markov chains.

3.1 Ornstein-Uhlenbeck process

Let X be an Ornstein-Uhlenbeck diffusion with parameter $\gamma > 0$, i.e. the differential operator associated to X is

$$Af = \frac{1}{2} \frac{d^2}{dx^2} f - \gamma x \frac{d}{dx} f.$$

We consider the optimal stopping problem (*) for X with the reward function

$$G_n(x) = (x^+)^n, \quad n \in \mathbb{N}.$$

For general Lévy-driven Ornstein-Uhlenbeck processes it was proved in [6] that the optimal stopping rule is one-sided, but no explicit way for finding the boundary explicitly in general was given there. Here we want characterize the optimal boundary for the Ornstein-Uhlenbeck diffusion using our approach. We have

$$(\beta - A)x^n = x^{n-2} \left((\beta + \gamma n)x^2 - \frac{n(n-1)}{2} \right) =: \tilde{f}_n(x).$$

Using Itô's lemma it is seen that \tilde{f}_n indeed has the mean value property for X_T , i.e.,

$$x^n = \mathbb{E}_x \left(\tilde{f}_n(X_T) \right) \quad \forall x$$

Recall that for Ornstein-Uhlenbeck processes the speed measure m is given by

$$m(dx) = 2e^{-\gamma x^2} dx$$

and

$$\psi_\beta(x) = e^{\gamma x^2/2} \mathcal{D}_{-\beta/\gamma}(-x\sqrt{2\gamma}),$$

where \mathcal{D}_ν denotes the parabolic cylinder function, see [3, A1.24,II.7]. Defining $Q_n = Q$ as in (21) we obtain

$$x^n = \mathbb{E}_x \left(Q_n(M_T) \right).$$

Using arguments similar to those in Lemma 2.13 it is easy to see that Q has a unique positive root x^* , which is given by the positive solution of the equation

$$\int_{-\infty}^x \tilde{f}(y) e^{-\gamma y^2/2} \mathcal{D}_{-\beta/\gamma}(-y\sqrt{2\gamma}) dy = 0. \quad (28)$$

An approximative value of x^* can be found using numerical methods. Notice that Q_n can also attain positive values on the negative half-line. However, this problem can be dealt with by considering the function

$$\hat{f}(x) = \begin{cases} 0, & x \leq 0, \\ Q(x), & x > 0, \end{cases}$$

which obviously satisfies the requirements of Theorem 2.4. Therefore, we obtain that the stopping time

$$\tau^* = \inf\{t : X_t \geq x^*\}$$

is optimal.

3.2 American put in a Lévy market

In this subsection we review the well known example of a perpetual American put option in a Lévy driven market in the light of our approach (cf. the solutions in [17] and [1], see also [5] for further discussions). Although the solution is well-known we want to demonstrate how to find the representing function \hat{f} systematically with the approach discussed in Subsection 2.2. Since the reward function $G(x) = K - e^x$, $K > 0$, is non-increasing we use the modified version of Theorem 2.4, see Remark 2.5. The American call option can be handled similarly.

The generator A of X is given by

$$Af(x) = \frac{c^2}{2} \frac{d^2}{dx^2} f(x) + b \frac{d}{dx} f(x) + \int_{\mathbb{R}} \left(f(x+y) - f(x) - y \frac{d}{dx} f(x) 1_{\{|y|<1\}} \right) \pi(dy), \quad (29)$$

where (b, c, π) is the Lévy triple of X and f is assumed to be in the domain of A . We wish to apply $\beta - A$ to $G(x) = K - e^x$. For this ansatz to work we assume that

$$\mathbb{E}_0(e^{X_1}) < e^\beta. \quad (30)$$

At the end we will see that the formula obtained is a solution for general X . We have

$$\begin{aligned} (\beta - A)G(x) &= \beta(K - e^x) + e^x \left(\frac{c^2}{2} + b + \int_{\mathbb{R}} (e^y - 1 - y 1_{|y|<1}) \pi(dy) \right) \\ &= \beta K - (\beta - \psi(1))e^x =: \tilde{f}(x), \end{aligned}$$

where $\psi(1) = \log \mathbb{E}_0(e^{X_1}) < \beta$. To find \hat{f} we calculate, cf. Proposition 2.10,

$$Q(x) = \frac{1}{\beta} \mathbb{E}_0(\tilde{f}(x + M_T)) = K - \frac{\beta - \psi(1)}{\beta} \mathbb{E}_0(e^{M_T}) e^x.$$

By noting that

$$\frac{\beta - \psi(1)}{\beta} = \frac{1}{\mathbb{E}_0(e^{X_T})} \quad (31)$$

and applying the Wiener-Hopf factorization (3) we obtain

$$Q(x) = K - \mathbb{E}_0(e^{I_T})^{-1} e^x.$$

Now we have a candidate for \hat{f} by our ansatz, i.e., $\hat{f} = Q \vee 0$, and formula (31) also makes sense for general Lévy processes without the integrability assumption (30). Indeed, one immediately sees that for general X we have

$$K - e^x = \mathbb{E}_x(Q(I_T)).$$

We can apply Theorem 2.4 (together with Remark 2.5) to \hat{f} and obtain that $x^* = \log(K\mathbb{E}_0(e^{I_T}))$ is the optimal stopping boundary and the value function is given by

$$V(x) = \mathbb{E}_x \left(\hat{f}(I_T) 1_{\{I_T < x^*\}} \right) = \mathbb{E}_x \left(\left(K - \frac{e^{I_T}}{\mathbb{E}_0(e^{I_T})} \right)^+ \right).$$

3.3 Novikov-Shiryaev optimal stopping problem

In [19] Novikov and Shiryaev solved the optimal stopping problem (*) with $G_n(x) = (x^+)^n$, $n = 1, 2, \dots$, for random walks and in [15] Kyprianou and Surya found the solution for Lévy processes. Here we discuss shortly how the solution can be found with the methodology presented in this paper. For simplicity, we assume that $\beta > 0$.

Let X be a Lévy process with the generator A as given in (29). We fix the exponent $n \in \{1, 2, \dots\}$ for the reward function G_n and assume that the Lévy measure π satisfies

$$\int_{(-\infty, -1) \cup (1, +\infty)} |y|^n \pi(dy) < \infty. \quad (32)$$

This condition implies that $\mathbb{E}_x(|X_t|^k) < \infty$ for all x , $t \geq 0$, and $k = 1, 2, \dots, n$. Using the independence of increments, it is easily seen that $\mathbb{E}_x(X_t^n)$ is a polynomial in t . Consequently, also the moments of X_T up to n exist.

Remark 3.1. *In [15] the solution is found under the weaker condition that (only) the integral over $(1, +\infty)$ in (32) is finite. Under this weaker condition the moments of M_T up to n exist but the moments of I_T may not exist. Working under the stronger condition reveals the rôle of the Appell polynomials of X_T which we find interesting and important. In fact, since the function Q_n , see (35), that is, the Appell polynomial associated with M_T , found in this way satisfies conditions (a) and (b) in Theorem 2.4 it is clear that our solution is indeed valid under the weaker condition and, hence, coincides with the solution in by Novikov and Shiryaev.*

We follow the receipt in Section 2.2 and operate with $\beta - A$ on x^n . This yields

$$\begin{aligned} \tilde{f}_n(x) &:= (\beta - A)(x^n) \\ &= \beta x^n - \left(\frac{c^2}{2} n(n-1) x^{n-2} + b n x^{n-1} \right) \\ &\quad - \int_{\mathbb{R}} \left((x+y)^n - x^n - y n x^{n-1} 1_{\{|y| \leq 1\}} \right) \pi(dy), \end{aligned}$$

where the integral term is well defined due to (32). Clearly, \tilde{f}_n is an n th order polynomial and satisfies the DE

$$\frac{d}{dx}\tilde{f}_n(x) = n\tilde{f}_{n-1}(x) \quad (33)$$

with $f_0 = 1$. We claim that

$$\mathbb{E}_x\left(\tilde{f}_n(X_T)\right) = \mathbb{E}_0\left(\tilde{f}_n(X_T + x)\right) = \beta x^n, \quad (34)$$

which, then, together with (33) shows that $x \mapsto \tilde{f}_n(x)/\beta$ is the n th order Appell polynomial associated with X_T , (for properties of the Appell polynomials, see [19] and [23]). To prove (33) we use standard computations, legitimate by (32), with the Itô formula for Lévy processes, see, e.g., [12, p. 612], to obtain

$$\begin{aligned} e^{-\beta t}\mathbb{E}_x(X_t^n) - x^n &= \mathbb{E}_x\left(\int_0^t e^{-\beta s}\left[\frac{c^2}{2}n(n-1)X_s^{n-2} + bnX_s^{n-1} - \beta X_s^n\right] ds\right) \\ &\quad + \mathbb{E}_x\left(\int_0^t e^{-\beta s}\int_{\mathbb{R}}[(X_s + y)^n - X_s^n - ynX_s^{n-1}1_{\{|y|\leq 1\}}]\pi(dy) ds\right). \end{aligned}$$

Multiplying both sides with β , letting $t \rightarrow \infty$ and using that the moments of X_t are polynomials in t we obtain

$$\begin{aligned} -\beta x^n &= \mathbb{E}_x\left(\int_0^\infty \beta e^{-\beta s}\left[\frac{c^2}{2}n(n-1)X_s^{n-2} + bnX_s^{n-1} - \beta X_s^n\right] ds\right) \\ &\quad + \mathbb{E}_x\left(\int_0^\infty \beta e^{-\beta s}\int_{\mathbb{R}}[(X_s + y)^n - X_s^n - ynX_s^{n-1}1_{\{|y|\leq 1\}}]\pi(dy) ds\right), \end{aligned}$$

which proves (34). Next we apply Proposition 2.10 to find the function Q_n and then \hat{f} . We write formula (18) therein as

$$Q_n(x) = \int_{-\infty}^0 Q_n^{(X_T)}(x+y)\mathbb{P}_0(I_T \in dy), \quad (35)$$

where $Q_n^{(X_T)}(x) := \tilde{f}_n(x)/\beta$. Recall from [23] the identity

$$Q_n^{(X_T)}(x+y) = \sum_{k=0}^n \binom{n}{k} Q_k^{(M_T)}(x) Q_{n-k}^{(I_T)}(y) \quad (36)$$

where $Q_i^{(M_T)}$ and $Q_i^{(I_T)}$ denote the i th Appell polynomials of M_T and I_T , respectively. Identity (36) follows easily from the Wiener-Hopf factorization

and the definition of the Appell polynomials. Substituting (36) into (35) and using the normalizations

$$Q_0^{(I_T)} \equiv 1 \quad \text{and} \quad \mathbb{E}_0\left(Q_i^{(I_T)}(I_T)\right) = 0 \text{ for all } i = 1, 2, \dots, n$$

yields $Q_n = Q_n^{(M_T)}$. Hence, our approach leads to the same characterization of the optimal stopping time in terms of $Q_n^{(M_T)}$ as in [19]. The crucial property to be able to apply Theorem 2.4, i.e., that $Q_n^{(M_T)}$ has a unique positive zero x^* and is non-decreasing for $x > x^*$, is proved in [19] Lemma 5. Consequently, it now easily follows that

$$\hat{f}(x) := \begin{cases} Q_n^{(M_T)}(x), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

satisfies conditions (a) and (b) in Theorem 2.4 and the solution of the optimal stopping problem results.

Note that Proposition 2.15 together with the fact that $x \mapsto \tilde{f}_n(x)/\beta$ is the n th order Appell polynomial associated with X_T reproduces the formula of the representing measure for spectrally positive Lévy processes obtained in [23, Corollary 8].

3.4 Absorbing Brownian motion

Let X be a Brownian motion on \mathbb{R}_+ absorbed at 0, i.e., $X_t = 0$ for $t \geq H_0$. We solve the optimal stopping problem (*) with $G_n(x) = x^n$, $n \geq 1$. The first step is to find \tilde{f}_n such that (13) holds. Guided by Subsection 2.2 we set

$$\tilde{f}_n(x) = \left(\beta - \frac{1}{2} \frac{d^2}{dx^2}\right) G_n(x) = x^{n-2} \left(\beta x^2 - \frac{n(n-1)}{2}\right) =: \beta P_n(x). \quad (37)$$

Instead of checking (13) via a direct calculation, we recall that P_n is the n th order Appell polynomial associated with a standard Brownian motion $(W_t)_{t \geq 0}$ started from 0 and evaluated at T , see [23, Example 3.9]. Hence, for all x

$$x^n = \mathbb{E}_0\left(P_n(x + W_T)\right) = \mathbb{E}_x\left(P_n(W_T)\right) = \frac{1}{\beta} \mathbb{E}_x\left(\tilde{f}_n(W_T)\right),$$

and, by the strong Markov property,

$$\begin{aligned} \mathbb{E}_x\left(\tilde{f}_n(W_T)\right) &= \mathbb{E}_x\left(\tilde{f}_n(W_T)1_{\{T < H_0\}}\right) + \mathbb{E}_x\left(\tilde{f}_n(W_T)1_{\{T > H_0\}}\right) \\ &= \mathbb{E}_x\left(\tilde{f}_n(X_T)1_{\{X_T > 0\}}\right) + \mathbb{E}_x\left(\mathbb{E}_x(\tilde{f}_n(W_T) \mid W_{H_0})1_{\{T > H_0\}}\right) \\ &= \mathbb{E}_x\left(\tilde{f}_n(X_T)1_{\{X_T > 0\}}\right). \end{aligned}$$

Consequently, we have

$$G_n(x) = \frac{1}{\beta} \mathbb{E}_x \left(\tilde{f}_n(X_T) 1_{\{X_T > 0\}} \right),$$

which is (13) if $n \neq 2$. To cover also the case $n = 2$ we abandon (13) and instead try to find a function Q_n such that

$$\mathbb{E}_x \left(\tilde{f}_n(X_T) 1_{\{X_T > 0\}} \right) = \mathbb{E}_x \left(Q_n(M_T) \right).$$

For this, the conditional distribution of X_T given M_T is needed, see (16). Since X is not conservative, this distribution has an atom at 0 and, hence, the formula in Lemma 2.11 is not valid. However, from [3, II.19] we may deduce

$$\mathbb{P}_x(M_T \in dz)/dz = \frac{\partial}{\partial z} \left(\frac{\psi_\beta(x)}{\psi_\beta(z)} \right)$$

and

$$\mathbb{P}_x(M_T \in dz, X_T \in dy) = 2 \frac{\psi_\beta(x) \psi_\beta(y)}{\psi_\beta^2(z)} dz dy.$$

where $\psi_\beta(x) = \text{sh}(x\sqrt{2\beta})$, $x \geq 0$. see *ibid.* p. 121. Consequently, the conditional density of X_T given M_T is given by

$$f_{X_T}(y|M_T = z) = \frac{2\psi_\beta(y)}{\psi'_\beta(z)}.$$

and, therefore,

$$\begin{aligned} Q_n(z) &:= \frac{1}{\beta} \int_0^z \tilde{f}_n(y) \frac{2\psi_\beta(y)}{\psi'_\beta(z)} dy \\ &= \frac{2}{\beta \sqrt{2\beta} \text{ch}(z\sqrt{2\beta})} \int_0^z y^{n-2} \left(\beta y^2 - \frac{n(n-1)}{2} \right) \text{sh}(y\sqrt{2\beta}) dy. \end{aligned}$$

Since the integrand changes its sign from $-$ to $+$ one can see (after some easy calculations) that Q_n satisfies (a) of Theorem 2.4. Although Proposition 2.12 is not directly applicable it can be proved analogously as therein that (b) of Theorem 2.4 holds for $\hat{f}_n := Q_n \vee 0$. Therefore, the optimal stopping boundary x^* is given as the unique positive solution to the equation

$$\int_0^z y^{n-2} \left(\beta y^2 - \frac{n(n-1)}{2} \right) \text{sh}(y\sqrt{2\beta}) dy = 0. \quad (38)$$

Integrating by parts, (38) is seen to be equivalent with

$$z \text{ch}(z\sqrt{2\beta}) - \frac{n}{\sqrt{2\beta}} \text{sh}(z\sqrt{2\beta}) = 0. \quad (39)$$

Notice that for $n = 1$ this equation does not have a positive solution and the optimal stopping rule is to stop immediately – in other words, reward $G(x) = x$ is β -excessive.

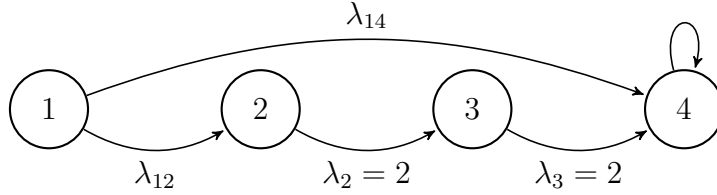
It is also interesting to observe that (39) is of the form

$$G_n(z) \frac{d\psi(z)}{dz} - \psi(z) \frac{dG_n(z)}{dz} = 0,$$

and, hence, our approach yields in this case the same characterization of the stopping point as given in Remark (ii) p. 97 in [22].

3.5 Markov chain

To see the general applicability of the theory we treat in this subsection an easy example with an underlying finite-state Markov chain. Let $(X_t)_{t \geq 0}$ be a continuous-time Markov chain with state space $E = \{1, \dots, 4\}$ and transition rates as given in the following figure.



To obtain explicit results let us assume that $\beta = 1$ and $G(1) = 3$, $G(2) = 1 = G(3)$ and $G(4) = 2$. We could use the method described in Subsection 2.2 to find \hat{f} such that $G(x) = \mathbb{E}_x(\sup_{t \leq T} \hat{f}(X_t))$ for all $x \in E$, but it may be instructive to find it directly: Since 4 is an absorbing state, we take $\hat{f}(4) = G(4) = 2$. To find $\hat{f}(3)$ we make the ansatz that $\hat{f}(3) \leq \hat{f}(4)$ and obtain

$$1 = G(3) = \mathbb{E}_3 \left(\sup_{t \leq T} \hat{f}(X_t) \right) = \hat{f}(4) \mathbb{P}_3(S_3 \leq T) + \hat{f}(3) \mathbb{P}_3(S_3 > T),$$

where the sojourn time S_3 at state 3 is exponentially distributed with parameter 2 and independent of T . Solving for $\hat{f}(3)$ yields

$$\hat{f}(3) = \frac{\lambda_3 + \beta}{\beta} \left(1 - 2 \frac{\lambda_3}{\lambda_3 + \beta} \right) = -1.$$

By a similar calculation we obtain that

$$\hat{f}(2) = \frac{(\beta + \lambda_2)^2}{\beta^2 + 2\beta\lambda_2} \left(1 - \hat{f}(4) \frac{\lambda_2^2}{(\beta + \lambda_2)^2} \right) = \frac{1}{5}.$$

Since 1 is a global maximum point of G taking

$$\hat{f}(1) = G(1) = 3$$

yields the right representation. We see that the assumptions of Theorem 2.6 are fulfilled (with $x_* = 2$, $x^* = 4$). Therefore

$$\tau^* = \inf\{t \geq 0 : X_t \neq 3\}$$

is an optimal stopping time.

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